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REMARKS ON THE FIRST MAIN THEOREM IN EQUIDISTRIBUTION THEORY. IV

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1. The purpose of this paper is to prove three theorems; their raison d'être will be given in § 5 where a formulation and discussion of some open problems of the subject will also be found. The first two theorems have to do with holomorphic mappings into C^n. Let us first recall some notation from Part III [16]. Let τ_0: $C^n \to R$ be $\tau_0 = \sum_i z_i\bar{z}_i$, and let

$$\omega_0 \equiv \frac{1}{4} dd^c\tau = \frac{\sqrt{-1}}{2} \sum_i dz_i \wedge d\bar{z}_i \,. \tag{1}$$

ω_0 is the Kähler form of the flat metric on C^n, whose volume element is

$$\Psi_0 = \frac{\omega_0^n}{n!} = \left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots dz_n \wedge d\bar{z}_n \,. \tag{2}$$

The first theorem is a derivative of Theorem 1 of Part III [16].

Theorem 1. *Let f: $C^n \to C^1$ be holomorphic such that df is nonsingular somewhere, and let $C_r^n = \{z\colon \sum_i z_i\bar{z}_i \leq e^r - 1\}$. Write $f = (f_1, \cdots, f_n)$. Then f is quasisurjective if*

(3)

$$\liminf_{r\to\infty} \frac{1}{\displaystyle\int_0^r dt \int_{C_t^n} \frac{f^*\omega_0^n}{(1 + \sum_i f_i\bar{f}_i)^{n+1}}} \int_{C_r^n} \frac{\omega_0 \wedge f^*\omega_0^{n-1}}{(1 + \sum_i z_i\bar{z}_i)(1 + \sum_i f_i\bar{f}_i)^{n-1}} = 0 \,.$$

There is a corollary to this theorem. Introduce the notation:

$$\begin{aligned} \omega_0 \wedge f^*\omega_0^{n-1} &= \frac{1}{n} \sigma_{n-1}^{\sharp}\omega_0^n \,, \\ f^*\omega_0^n &= \sigma_n^{\sharp}\omega_0^n \,. \end{aligned} \tag{4}$$

Then $\sigma_{n-1}^{\sharp}$ and $\sigma_n^{\sharp}$ are respectively the $(n-1)$-th and the n-th elementary

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symmetric functions of the matrix $\left\{\sum_k \frac{\partial f_k}{\partial z_i} \frac{\overline{\partial f_k}}{\partial z_j}\right\}$. In other words ,

(5)

$$\sigma^\#_{n-1} = \sum_i \sum_\sigma \varepsilon(\sigma) \left(\sum_k \frac{\partial f_k}{\partial z_1} \frac{\overline{\partial f_k}}{\partial z_{\sigma(1)}}\right) \cdots \widehat{\left(\sum_k \frac{\partial f_k}{\partial z_i} \frac{\overline{\partial f_k}}{\partial z_{\sigma(i)}}\right)} \cdots \left(\sum_k \frac{\partial f_k}{\partial z_n} \frac{\overline{\partial f_k}}{\partial z_{\sigma(n)}}\right),$$

$$\sigma^\#_n = \det \left\{\sum_k \frac{\partial f_k}{\partial z_i} \frac{\overline{\partial f_k}}{\partial z_j}\right\} = \left|\det \left(\frac{\partial f_i}{\partial z_j}\right)\right|^2,$$

where σ runs through all permutations of $\{1, \cdots, i-1, i+1, \cdots, n\}$, $\varepsilon(\sigma)$ is the sign of the permutation, and the roof $\wedge$ indicates that the corresponding factor is omitted.

Corollary. *For $f\colon C^n \to C^n$ as above, if, for some positive K,*

$$(6) \qquad \left(\frac{1}{n}\sigma^\#_{n-1}\right)^n \leq K \left(\frac{\left|\det\left(\frac{\partial f_i}{\partial z_j}\right)\right|^2}{1 + \sum_i f_i \bar{f}_i}\right)^{n-1} \cdot \frac{1}{1 + \sum_i z_i \bar{z}_i},$$

then f is quasisurjective.

When the domain manifold is replaced by a general V with a convex exhaustion, we have the somewhat weaker result:

Theorem 2. *Let $f\colon V \to C^n$ be holomorphic, where V has dimension n and a fixed convex exhaustion τ and df is nonsingular somewhere. Write $V[r] = \{p\colon \tau(p) \leq r\}$ as usual. Then f is quasisurjective if the following two conditions hold:*

$$(7) \qquad \liminf_{r\to\infty} \frac{1}{\int_0^r dt \int_{V[t]} f^*\Psi_0} \int_{V[r]} dd^c\tau \wedge f^*\omega_0^{n-1} = 0 ,$$

$$(8) \qquad \liminf_{r\to\infty} \frac{1}{\int_0^r dt \int_{V[t]} f^*\Psi_0} \int_{\partial V[r]} d^c\tau \wedge f^*(\tau_0\omega_0^{n-1}) = 0 .$$

For the third theorem, let V be as above, M an n-dimensional compact Kähler manifold and $f\colon V \to M$ a holomorphic mapping. Then the holomorphy of f implies that for each $a \in M$,

$$0 \leq n(r_1, a) \leq n(r_2, a) ,$$

if $r_1 \leq r_2$. So the following definition is meaningful:

$$(9) \qquad n(V, a) = \lim_{r\to\infty} n(r, a) .$$

Theorem 3. *Let $f: V \to M$ be holomorphic, where* $\dim V = \dim M$, *V admits a fixed convex exhaustion τ and M is compact Kählerian with Kähler form κ. Assume that $n(V, f(u))$ is a fixed finite constant n_0 for almost all $v \in V$. Then a necessary and sufficient condition for f to be quasisurjective is that for every C^2 function φ on M,*

$$(10) \qquad \lim_{r\to\infty} \frac{1}{T(r)} \int_{V[r]} df^*\varphi \wedge d^c\tau \wedge f^*\kappa^{n-1} = 0 .$$

If $f: V \to M$ is an *analytic cover* over $f(V)$, then the condition that $n(V, f(v)) = n_0$ for almost all $v \in V$ is met. In particular, every holomorphic imbedding f would do; such is the case of the Fatou-Bieberbach mapping. This theorem should be improved in two respects: the restriction on $n(V, f(v))$ should be removed and the intrinsic geometric structure of V and M should be invoked in place of the ring of C^2 functions on M.

2. This section proves Theorem 1 and its corollary. First recall that an $(n-1, n-1)$ form η is *positive* (Part I [12]) iff locally there exist $(n-1, 0)$ forms θ_i such that

$$\eta = (\sqrt{-1})^{\text{sign}} \sum_i \theta_i \wedge \bar{\theta}_i ,$$

where sign $= (n-1)^2$. Similarly, a $(1, 1)$ form ζ is *positive* iff locally there exist $(1, 0)$ forms ξ_i such that

$$\zeta = \sqrt{-1} \sum_i \xi_i \wedge \bar{\xi}_i .$$

Given two real (n, n) forms Φ and Λ, we write $\Phi \geq \Lambda$ iff $\Phi = \varphi\Psi_0$, $\Lambda = \lambda\Psi_0$ and the real functions φ and λ satisfy $\varphi \geq \lambda$. Note that if η is a positive $(n-1, n-1)$ form, and ζ, ζ' are positive $(1, 1)$ forms, then clearly: (1) $\eta\wedge\zeta' \geq 0$, (2) $\zeta^{(n-1)}$ is positive, and (3) $\eta\wedge(\zeta + \zeta') \geq \eta\wedge\zeta$.

Now the Kähler form of the Fubini-Study metric on C^n is (Part III [16], §2):

$$(11) \qquad \omega_1 = \frac{\sqrt{-1}}{2} \frac{\sum_i dz_i \wedge d\bar{z}_i}{(1 + \sum_i z_i\bar{z}_i)} - \frac{\sqrt{-1}}{2} \frac{\sum_i \bar{z}_i dz_i \wedge \sum_i z_i d\bar{z}_i}{(1 + \sum_i z_i\bar{z}_i)^2} ,$$

and the volume element is

$$(12) \qquad \Psi_1 = \frac{\omega_1^n}{n!} = \frac{1}{(1 + \sum_i z_i\bar{z}_i)^{n+1}} \Psi_0 .$$

Furthermore,

$$\int_{C^n} \Psi_1 = \frac{\pi^n}{n!} \,. \tag{13}$$

In view of (1), we may rewrite (11) as

$$\omega_1 = \frac{\omega_0}{(1 + \sum_i z_i \bar{z}_i)} - \frac{1}{2} d'\tau_1 \wedge \overline{d'\tau_1} \,.$$

where $\tau_1 = \log(1 + \sum_i z_i \bar{z}_i)$ as usual. ω_0, ω_1 as well as $d'\tau \wedge \overline{d'\tau_1}$ are positive (1, 1) forms. If $f: C^n \to C^n$ is holomorphic, then $f^*\omega_0$, $f^*\omega_1$ and $f^*(d'\tau_1 \wedge \overline{d'\tau_1})$ are also positive. Therefore, the remarks above imply that

$$\begin{aligned} \omega_1 \wedge (f^*\omega_1)^{n-1} &\leq \frac{\omega_0}{(1 + \sum_i z_i \bar{z}_i)} \wedge (f^*\omega_1)^{n-1} \\ &\leq \frac{\omega_0}{(1 + \sum_i z_i \bar{z}_i)} \wedge \frac{f^*\omega_0}{(1 + \sum_i f_i \bar{f}_i)} \wedge (f^*\omega_0)^{n-2} \\ &\leq \cdots \\ &\leq \frac{\omega_0}{(1 + \sum_i z_i \bar{z}_i)} \wedge \frac{(f^*\omega_0)^{n-1}}{(1 + \sum_i f_i \bar{f}_i)^{n-1}} \,. \end{aligned}$$

By Theorem 1 of Part III [16], $f: C^n \to C^n \subset P_n C$ is quasisurjective if

$$\liminf_{r \to \infty} \frac{\int_{C_r^n} \omega_1 \wedge f^*\omega_1^{n-1}}{\int_0^r dt \int_{C_t^n} f^*\Psi_1} = 0 \,. \tag{14}$$

Hence, if (3) holds, the above inequality together with (12) will imply that (14) is satisfied. This proves Theorem 1.

The proof of the corollary is patterned after the proof of Theorem 3 of Part III. We wish to show that (6) entails (3). To this end, first observe that (6) is equivalent to:

$$\left\{ \frac{\sigma_{n-1}^\sharp (1 + \sum_i z_i \bar{z}_i)^n}{n(1 + \sum_i f_i \bar{f}_i)^{n-1}} \right\}^{n/(n-1)} \leq K^{1/(n-1)} \sigma_n^\sharp \left\{ \frac{1 + \sum_i z_i \bar{z}_i}{1 + \sum_i f_i \bar{f}_i} \right\}^{n+1} . \tag{15}$$

For brevity, let us denote the left side of (15) by $\varphi^{n/(n-1)}$, and the right side by $K^{1/(n-1)} \Psi$. Then (4) and (12) imply that

$$\frac{1}{\displaystyle\int_0^r dt \int_{C_t^n} \frac{f^*\omega_0^n}{(1+\sum_i f_i\bar{f}_i)^{n+1}}} \int_{C_r^n} \frac{\omega_0 \wedge f^*\omega_0^{n-1}}{(1+\sum_i z_i\bar{z}_i)(1+\sum_i f_i\bar{f}^{n-1})}$$

$$= \frac{\displaystyle\int_{C_r^n} \varphi\Psi_1}{\displaystyle\int_0^r dt \int_{C_t^n} \phi\Psi_1}$$

$$\leq \frac{\left(\displaystyle\int_{C_r^n} \varphi^{n/(n-1)}\Psi_1\right)^{(n-1)/n} \left(\displaystyle\int_{C_r^n} \Psi_1\right)^{1/n}}{\displaystyle\int_0^r dt \int_{C_t^n} \phi\Psi_1} \qquad \text{(Hölder's Inequality)}$$

$$\leq \text{const.} \frac{\left(\displaystyle\int_{C_r^n} \varphi^{n/(n-1)}\Psi_1\right)^{(n-1)/n}}{\displaystyle\int_0^r dt \int_{C_t^n} \phi\Psi_1} \qquad (13)$$

$$\leq \text{const.} \frac{\left(\displaystyle\int_{C_r^n} \phi\Psi_1\right)^{(n-1)n}}{\displaystyle\int_0^r dt \int_{C_t^n} \phi\Psi_1}. \qquad (15)$$

Let $g(t) = \left(\displaystyle\int_{C_t^n} \varphi\Psi_1\right)^{(n-1)/n}$. To prove (3), it suffices to prove:

$$\liminf_{r\to\infty} \frac{g(r)}{\displaystyle\int_0^r g(t)^{n/(n-1)} dt} = 0 .$$

Let $G(r) = \displaystyle\int_0^r g(t)^{n/(n-1} dt$. Then it is equivalent to proving:

$$\liminf_{r\to\infty} \frac{G'(r)}{G(r)^{n/(n-1)}} = 0 .$$

By a well-known lemma (Cf. Lemma 7.2 of [8]), if $1 < k < n/(n-1)$, then $G'(r) \leq G^k(r)$ on an infinite sequence of real numbers diverging to infinity. Hence,

$$0 \leq \liminf_{r\to\infty} \frac{G'(r)}{G(r)^{n/(n-1)}} \leq \liminf_{r\to\infty} \frac{1}{G(r)^{(n/(n-1))-k}} = 0 ,$$

because $n/(n-1) - K > 0$ and $G(r) \to +\infty$ as $r \to +\infty$. q.e.d.

Note finally that using the idea of the above proof and invoking Theorem 5.1 of Part II [13], we can get a result quite similar to Theorem 2 of this paper.

3. We now come to Theorem 2. The proof is a technical variation of that of the theorem in the Appendix of Part III [16], so it will only be outlined. On the other hand, there is a high probability of confusion in the symbols employed, so we shall be careful in this respect.

If $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$, let $\|z\| = (\sum_i z_i\bar{z}_i)^{1/2}$. Define

$$\xi_a = \frac{-1}{S_{2n}(2n-2)} \frac{1}{\|z-a\|^{2n-2}} \Psi_0 ,$$

where $S_{2n} = \dfrac{2\pi^n}{(n-1)!}$ is the volume of the unit sphere in $\boldsymbol{C}^n$. If $\Delta = \dfrac{-1}{4}\sum_i \dfrac{\partial^2}{\partial z_i \partial \bar{z}_i}$ is the Laplacian of the flat metric of $\boldsymbol{C}^n$, then it is classical that

$$(16) \qquad \Delta \xi_a = -\delta_a ,$$

where δ_a is the Dirac measure at a as usual. If

$$\psi = -\frac{4}{n}(\sum_i z_i\bar{z}_i)\Psi_0 = -\frac{4}{n}\tau_0\Psi_0 ,$$

and $\pi_a = \psi + \xi_a$, then

$$(17) \qquad \Delta \pi_a = \Psi_0 - \delta_a .$$

Introduce the notation:

$$\widehat{dz_i} = (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (\widehat{dz_i} \wedge d\bar{z}_i) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n) ,$$

$$\widehat{d\bar{z}_i} = (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_i \wedge \widehat{d\bar{z}_i}) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n) .$$

If we denote by $\mu_a = \delta\pi_a = -{*d*}\pi_a$, then

$$(18) \qquad \mu_a = \delta\psi + \frac{1}{S_{2n}}\left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{\|z-a\|^{2n}} \sum_i \overline{(z_i - a_i)}\widehat{d\bar{z}_i} - (z_i - a_i)\widehat{dz_i} .$$

Since $d\mu_a = d\delta\pi_a = \Delta\pi_a = \Psi_0 - \delta_a$, we have

$$d\mu_a = \Psi_0 \quad \text{in} \quad C^n - \{a\} . \tag{19}$$

Let $\lambda_a = -\Lambda\pi_a$; then $\delta = \Lambda d^c - d^c\Lambda$ implies

$$\mu_a = d^c\lambda_a , \tag{20}$$

$$\lambda_a = \frac{4}{(n-1)!n}\tau_0\omega_0^{n-1} + \frac{1}{4(n-1)\pi^n}\frac{1}{\|z-a\|^{2n-2}}\omega_0^{n-1} . \tag{21}$$

So clearly λ_a is a positive $(n-1, n-1)$ form.

Suppose D is a compact complex manifold with boundary, and $f\colon D \to C^n$ is holomorphic. Let $a \in C^n$, and $f^{-1}(a)$ be finite and disjoint from ∂D. Then (18) – (20) lead to

$$\int_D f^*\Psi_0 = n(D, a) + \int_{\partial D} d^c f^*\lambda_a . \tag{22}$$

(Cf. the proof of equation (21) in Part III [16].) The rest is familiar by now. Let $f\colon V \to C^n$ be holomorphic as in Theorem 1, and let a be a regular value of f. The reasoning in §2 of Part II [13] leads to

$$\int_{r_0}^{r} n(t, a)dt = \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a + \left(\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a + \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau \right) . \tag{23}$$

If f is *not* quasisurjective, let E be a bounded measurable subset in $C^n - f(V)$ such that $\int_E \Psi_0 = \varepsilon > 0$. We now integrate (23) with respect to a and Ψ_0 over E. At this point, we again refer the reader to Part II [13, §§4–5] for a discussion of how the positivity of $dd^c\tau$ and λ_a and Fubini's Theorem justify the interchanges of order of integration in the sequel. Since $n(t, a) = 0$ for all t and for all $a \in E$,

$$\int_E \Psi_0 \int_{r_0}^{r} n(t, a)dt = \int_{r_0}^{r} dt \left(\int_E n(t, a)\Psi_0 \right) = 0 .$$

Thus (23) leads to

$$0 = \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_E \Psi_0 \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a + \int_E \Psi_0 \left(\int_{\partial V[r_0]} d^c\tau \wedge f^*\lambda_a + \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau \right)$$

$$\geq \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_E \Psi_0 \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a \,,$$

because the other term is positive. Since E is bounded, we may assume it to be contained in the closed ball B of finite radius ρ in C^n. Furthermore, $\int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a$ is a positive function of a, so that

$$\begin{aligned} 0 &\geq \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_E \Psi_0 \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a \\ &\geq \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_B \Psi_0 \int_{\partial V[r]} d^c\tau \wedge f^*\lambda_a \\ &= \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_{\partial V[r]} d^c\tau \wedge f^*\omega_0^{n-1} \\ &\quad \times f^*\left(\int_B \left(\frac{4\tau_0}{(n-1)!n} + \frac{1}{4(n-1)\pi^n} \frac{1}{\|z-a\|^{2n-2}}\right)\Psi_0(a)\right) \end{aligned}$$

by virtue of (21). Now,

$$\int_B \frac{4\tau_0}{n(n-1)!} \Psi_0(a) = C\tau_0 \,,$$

$$\int_B \frac{1}{4(n-1)\pi^n} \frac{1}{\|z-a\|^{2n-1}} \Psi_0(a) \leq C' \,,$$

where C and C' are finite constants, which depend on ρ but are independent of r and a. (Cf. Lemma in the Appendix of Part III [16].) So the above inequality becomes

$$\begin{aligned} 0 &\geq \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - \int_{\partial V[r]} f^*(C\tau_0 + C')d^c\tau \wedge f^*\omega_0^{n-1} \\ &= \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - C \int_{\partial V[r]} d^c\tau \wedge f^*(\tau_0\omega_0^{n-1}) - C' \int_{\partial V[r]} d^c\tau \wedge f^*\omega_0^{n-1} \\ &= \varepsilon \int_{r_0}^{r} dt \int_{V[t]} f^*\Psi_0 - C \int_{\partial V[r]} d^c\tau \wedge f^*(\tau_0\omega_0^{n-1}) - C' \int_{V[r]} dd^c\tau \wedge f^*\omega_0^{n-1} \,, \end{aligned}$$

where the last step is due to Stokes' theorem and the fact that $d\omega_0^{n-1} = 0$. Thus, if (7) and (8) are true, then the above inequalities would imply $0 \geq \varepsilon > 0$, a contradiction. Therefore f is quasisurjective.

4. Before giving the proof of Theorem 3, we make some general comments. Let $f: V \rightarrow M$ be a holomorphic mapping between complex manifolds of the same dimension as usual, and first recall equation (4) in §2 of Part II [13]:

$$(24)\qquad N(r,a) = T(r) + \int_{V[r]-V[r_0]} f^*\lambda_a \wedge dd^c\tau - \int_{\partial V[t]} d^c\tau \wedge f^*\lambda_a \Big|_{t=r_0}^{t=r} .$$

It is necessary first to elaborate on the form λ_a. From §2 of Part I [12], we know that $\lambda_a = (g(a,\cdot) + K)\Lambda\Psi$, where g is the kernel of the Green operator G, and K is a constant so chosen that $g + K > 0$ on $M \times M$ (see equation (9) of [12]). If ρ is any continuous function on M such that $\int_M \rho\Psi = 1$, then

$$(25)\qquad \int_M (g(a,y) + K)\rho(a)\Psi(a) = (G\rho)(y) + K .$$

Note that $G\rho$ becomes a C^1 function (p. 157 of de Rham [5]). Suppose now we integrate (24) with respect to $\rho\Psi$ and a over M. Always assuming that V has a convex exhaustion, we have, by (25),

$$\begin{aligned}
&\int_{r_0}^{r} dt \int_M n(t,a)\rho(a)\Psi(a) \\
&\quad = T(r) + \int_{V[r]-V[r_0]} dd^c\tau \wedge f^*\Lambda\Psi \cdot f^*\Big(\int_M (g(a,\ \cdot) + K)\rho\Psi\Big) \\
&\qquad - \int_{\partial V[t]} d^c\tau \wedge f^*\Lambda\Psi \cdot f^*\Big(\int_M (g(a,\cdot) + K)\rho\Psi\Big)\Big|_{r_0}^{r} \\
&\quad = T(r) + \int_{V[r]-V[r_0]} f^*(G\rho + K) dd^c\tau \wedge f^*\Lambda\Psi - \int_{\partial V[t]} f^*(G\rho + K) d^c\tau \wedge f^*\Lambda\Psi \Big|_{r_0}^{r} \\
&\quad = T(r) + \int_{V[r]-V[r_0]} (f^*G\rho) dd^c\tau \wedge f^*\Lambda\Psi - \int_{\partial V[t]} (f^*G\rho) d^c\tau \wedge f^*\Lambda\Psi \Big|_{r_0}^{r} \\
&\qquad + K\Big\{ \int_{V[r]-V[r_0]} dd^c\tau \wedge f^*\Lambda\Psi - \int_{\partial V[t]} d^c\tau \wedge f^*\Lambda\Psi \Big|_{r_0}^{r} \Big\} .
\end{aligned}$$

But $K\{\quad\} = 0$ because of Stokes' theorem and the fact that $\Lambda\Psi$ is a closed form. As $f^*G\rho$ is C^1, we may apply Stokes' theorem once more so that

$$\begin{aligned}
\int_{r_0}^{r} dt \int_M n(t,a)\rho(a)\Psi(a) &= T(r) + \int_{V[r]-V[r_0]} (f^*G\rho) dd^c\tau \wedge f^*\Lambda\Psi \\
&\quad - \int_{V[r]-V[r_0]} d((f^*G\rho) d^c\tau \wedge f^*\Lambda\Psi) \\
&= T(r) - \int_{V[r]-V[r_0]} d(f^*G\rho) \wedge d^c\tau \wedge f^*\Lambda\Psi .
\end{aligned}$$

So finally we obtain

(26)

$$1 - \frac{1}{T(r)} \int_{r_0}^{r} dt \int_M n(t, a)\rho(a)\Psi(a) = \frac{1}{T(r)} \int_{V[r]-V[r_0]} d(f^*G\rho)\wedge d^c\tau\wedge f^*\Lambda\Psi$$

only under the assumption that ρ is continuous and that $\int_M \sigma\Psi = 1$. This is the basic formula we will use.

Now we come to the proof of Theorem 3. First assume f is quasisurjective. By virtue of L'Hospital's Rule and Lemma 6.2 of [8],

$$\lim_{r\to\infty} \frac{\int_{r_0}^{r} dt \int_M n(t, a)\rho(a)\Psi(a)}{T(r)} = \lim_{r\to\infty} \frac{\int_M n(r, a)\rho(a)\Psi(a)}{\int_{V[r]} f^*\Psi}$$

$$= \lim_{r\to\infty} \frac{\int_M n(r, a)\rho(a)\Psi(a)}{\int_M n(r, a)\Psi(a)} .$$

Since $M - f(V)$ is a set of measure zero and, by assumption, $n(V, f(v)) = n_0$ for almost all $v \in V$, we have that $\lim_{r\to\infty} n(r, a) = n_0$ a.e. on M. Hence Lebesque's bounded convergence theorem implies that the above limit equals

$$\frac{\int_M \lim_{r\to\infty} n(r, a)\rho(a)\Psi(a)}{\int_M \lim_{r\to\infty} n(r, a)\Psi(a)} = \frac{n_0 \int_M \rho\Psi}{n_0 \int_M \Psi} = 1 .$$

With this, (26) becomes

$$\lim_{r\to\infty} \frac{1}{T(r)} \int_{V[r]} d(f^*G\rho)\wedge d^c\tau\wedge f^*\Lambda\Psi = 0 \tag{27}$$

for all continuous ρ such that $\int_M \rho M = 1$. If now φ is a given C^2 function on M, then $\Delta\varphi$ is a continuous function. In particular, $\int_M (\Delta\varphi + 1)\Psi = 1$ because $\int_M (\Delta\varphi)\Psi = 0$. Thus for $\rho = \Delta\varphi + 1$, (27) holds. But $G\Delta\varphi + c = \varphi$, where

$c = \int_M \varphi\Psi$ (p. 157 of de Rham [5]). Furthermore, $G1 = 0$. Hence $G\rho = G(\Delta\varphi + 1) = \varphi - c$ and $d(f^*G\rho) = df^*\varphi$, which together with $\Lambda\Psi = \frac{1}{n!}\kappa^{n-1}$ prove that (10) holds.

Next, suppose (10) holds for every C^2 function φ on M, and we must prove that f is quasisurjective provided $n(V, f(v)) = n_0$ for almost all $v \in V$. Suppose f is not quasisurjective; then $\int_{M-f(V)} \Psi = \varepsilon > 0$. Since n_0 is finite, $f(V)$ must have interior. Let φ be a C^∞ function with support in $f(V)$ such that $\int_M \varphi\Psi = 1$. As before,

$$\lim_{r\to\infty} \frac{\int_{r_0}^{r} dt \int_M n(t, a)\varphi(a)\Psi(a)}{T(r)} = \lim_{r\to\infty} \frac{\int_M n(r, a)\varphi(a)\Psi(a)}{\int_M n(r, a)\Psi}.$$

Let χ be the characteristic function of the set $f(V)$ in M. Then our assumptions imply that $\lim_{r\to\infty} n(r, a) = n_0\chi$ and $\int_M \chi\Psi = (1 - \varepsilon)$. So Lebesque's bounded convergence theorem again implies that the above limit equals

$$\frac{\int_M \lim_{r\to\infty} n(r, a)\varphi(a)\Psi(a)}{\int_M \lim_{r\to\infty} n(r, a)\Psi} = \frac{\int_M \chi\varphi\Psi}{\int_M \chi\Psi} = \frac{1}{1-\varepsilon},$$

because support $\varphi \subseteq f(V)$ implies $\chi\varphi = \varphi$, and $\int_M \varphi\Psi = 1$. So replacing ρ by φ in (26), we have

$$\lim_{r\to\infty} \frac{1}{T(r)} \int_{V[r]} df^*G\varphi \wedge d^c\tau \wedge f^*\Lambda\Psi = 1 - \frac{1}{1-\varepsilon} = -\varepsilon < 0,$$

which contradicts (10) because $G\varphi$ is C^∞.

5. In this section, we shall discuss some open problems in n-dimensional equidistribution theory which seem to be of significance. In accordance with the precedence set in Part I [12], the discussion will be centered exclusively on the equi-dimensional case. So let V, M be complex manifolds of dimension n, V open and M compact, and let $\mathscr{H}(V, M)$ be the space of holomorphic mappings from V into M equipped with the *compact open topology* (Cf. e.g.

[14, §1]). In general, it is far from clear that $\mathscr{H}(V, M)$ has any member whose differential is somewhere nonsingular even if we assume V to have a convex exhaustion and M to be compact Kählerian. We shall therefore even limit ourself to the case $V = C^n$ and $M = P_nC$ in the following.

So denote $\mathscr{H}(C^n, P_nC)$ simply by $\mathscr{H}$, and let Q stand for the subset in $\mathscr{H}$ of all the quasisurjective mappings. One of the central objects of the study in equidistribution theory is the "structure" of Q and $\mathscr{H} - Q$. More precisely:

Problem 1. Q has no interior and possesses isolated points.

Problem 2. Characterize the isolated points of Q.

Problem 3. $\mathscr{H} - Q$ has interior.

Problem 4. There are points $f \in \mathscr{H} - Q$ such that every neighborhood of f in $\mathscr{H}$ contains points of Q.

Problem 5. Describe $\mathscr{H}$, Q and $\mathscr{H} - Q$ in terms of arcwise-connectivety.

The above problems are admittedly difficult. As a first step, one might consider the easier problems:

Problem 6. Construct a sequence $\{f_n\}$ such that f_n converges to the inclusion mapping $C^n \subseteq P_nC$ but $\{f_n\} \subseteq \mathscr{H} - Q$.

Problem 7. Is the Fatou-Bieberbach mapping $F: C^2 \to C^2 \subseteq P_2C$ (Bochner-Martin [15, p. 45]) an interier point of $\mathscr{H} - Q$?

Real progress will be made if one can get some insight into the Fatou-Bieberbach mapping F. Recall that it is a holomorphic imbedding of C^2 into C^2, which omits a neighborhood of $(1, 1)$. Since $F(C^2)$ must be contractible, F has to omit a much larger subset of C^2. So one of the problems is to give a precise description of $F(C)$. In particular, describe the inverse image under F of a sufficiently large neighborhood of $(1, 1)$. Another approach to the same problem is to prove a necessary condition for quasisurjectivity, and then to prove *directly* that F violates that condition. Theorem 3 of this paper arose from such an attempt and is in fact applicable to F. But as explained above, the fact that it involves the ring of C^2 functions is not satisfactory. So one of the urgent problems is to decide whether the converse of Theorem 1 of Part III [16] is valid. In other words, we have

Problem 8. If $f: C^n \to P_nC$ is quasisurjective, then necessarily

$$\liminf_{r \to \infty} \frac{\displaystyle\int_{C_r^n} \omega_1 \wedge f^*\kappa^{n-1}}{\displaystyle\int_0^r dt \int_{C_t^n} f^*\kappa^n} = 0 .$$

An affirmative answer even for a holomorphic imbedding f would be extremely valuable.

This series of papers is essentially concerned with *sufficient* conditions for quasisurjectivity. One of the main problems in this direction is to insure quasisurjectivity in terms of function theoretic properties of the holomorphic mapping. In this light, such sufficient conditions are useless unles one can apply them to concrete cases. This explains the concern with mappings C^n rather than into P_nC and the replacement of the Fubini-Study metric by the flat metric of C^n, (Appendix of Part III [16], and Theorems 1 and 2 of this paper), for the Fubini-Study metric is computationally untractable. The Corollary to Theorem 1 of this paper is a weak result, but represents a first step in this direction. Among other things, I would like to pose the following three problems:

Problem 9. Every holomorphic function $\varphi: C^n \to C$ gives rise to a holomorphic mapping $f_\varphi: C^n \to C^n$ by $(\partial\varphi/\partial z_1, \cdots, \partial\varphi/\partial z_n)$ (Gauss map). When is f_φ quasisurjective?

Problem 10. Let $\{f_{ij}\}$ be n^2 holomorphic functions of a single variable. Are nondegenerate holomorphic mappings of the type $(f_{11}(z_1)\cdots f_{1n}(z_n), \cdots, f_{n1}(z_1)\cdots f_{nn}(z_n))$ quasisurjective?

Problem 11. If $f: C^n \to C^n$ is quasiconformal with respect to the flat metrics, then f is quasisurjective.

Note that quasiconformality with respect to flat metrics can be expressed in purely function theoretic terms. See [14, Appendix].

The following problems are perhaps of secondary importance, but we list them because they still seem to be of some interest.

Problem 12. $f \in \mathscr{H} - Q$ iff it omits an open set.

Problem 13. Are there *nonlinear* quasiconformal mappings of $C^n \to C^n$ with respect to the flat metrics?

Problem 14. Construct *nonlinear* quasiconformal mappings of $C^n \to P_nC$ with respect to the Fubini-Study metrics.

As a background for Problem 14, one should recall that such mappings are always quasisurjective (Corollary 4, Part III [16]) and note that every linear map $C^n \to C^n \subseteq P_nC$ is necessarily quasiconformal with respect to the Fubini-Study metrics.

Finally, it remains to note that the compact open topology may not be the "right" topology to put on $\mathscr{H}$ from the point of view of equidistributions. Let us denote by $\mathscr{H}^*$ the set of holomorphic mappings from $C^n \to P_nC$ equipped with the *uniform topology*, i.e., a typical neighborhood of $f: C_n \to P_nC$ is $\{g: \sup_{p \in C^n} \text{dist}\,(f(p), g(p)) < \varepsilon\}$, where $\varepsilon > 0$ and dist (,) is the global distance function of the Fubini-Study metric. Note that convergence in this topology implies uniform convergence of derivatives of all orders. Regarding this $\mathscr{H}^*$, we have the following theorem by virtue of Theorem 2 and Corollary 4 of Part III [16].

Theorem 4. *If $f: C^n \to P_nC$ is either of bounded distortion or quasiconformal with respect to the Fubini-Study metrics, then there is a neighborhood of f in $\mathscr{H}^*$ consisting of quasisurjective mappings.*

On the other hand, if $f: C^n \to P_n C$ omits an open set (e.g. Fatou-Bieberbach map), then clearly some neighborhood of f in $\mathscr{H}^*$ will consist of mappings omitting an open set. Thus, if Q denotes the quasisurjective members of $\mathscr{H}^*$, then both Q and $\mathscr{H}^* - Q$ will have interior.

Problem 15. Q has isolated points.

Problem 16. Give necessary and sufficient conditions for $f \in Q$ to be an interior point.

Problem 17. Describe $\mathscr{H}^*$, Q and $\mathscr{H}^* - Q$ in terms of arcwise-connectivity.

Now if Problem 12 should have an affirmative solution, then every point of $\mathscr{H}^* - Q$ will be an interior point so that $\mathscr{H}^* - Q$ will be open in $\mathscr{H}^*$. On the basis of this, let us pose a very vague question:

Problem 18. Can one express in a quantitative way the fact that Q is a "smaller" set than $\mathscr{H}^* - Q$?

References

(*Continuation of Part* III [16])

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